

# The Curvature of the Hitchin Connection

Jørgen Ellegaard Andersen and Niccolo Skovgård Poulsen

September 6, 2016

## Abstract

In this paper we calculate the curvature of the Hitchin connection. We further show that a slight (possibly trivial) modification of the Hitchin connection has curvature equal to an explicit given multiple of the Weil-Petersen symplectic form on Teichmüller space.

Dedicated to Nigel Hitchin at the conference *Hitchin70*, celebrating his 70'th Birthday.

## 1 Introduction

In [10] Hitchin introduce a projectively flat connection in the bundle of quantizations of the moduli spaces  $M$  of flat  $SU(n)$ -connections over a surface of genus  $g > 1$  with central holonomy around a marked point on the surface. This connection was also constructed in [7] by Axelrod, Della Pietra and Witten from a more physical perspective, where it was also established how it is related to quantum Chern-Simons theory. See also [3], where it was shown how these two constructions agree and can be slightly generalised. Let us here briefly recall the setup.

The moduli space  $M$  is compact and smooth in the co-prime case, i.e. in case when the central holonomy around the special marked point generates the centre of  $SU(n)$ . In general it has a smooth part  $M'$ , which consists of the irreducible connections (if  $n = 2$ , then  $g > 2$  for this to be the case, since  $M = \mathbb{P}^3$  in the case  $(g, n) = (2, 2)$ ). The smooth part  $M'$  has a natural symplectic form called the Seshadri-Atiyah-Bott-Goldmann symplectic form. The Chern-Simons line bundle  $\mathcal{L}$  over  $M$  is a prequantum line bundle for  $\omega$  [8]. By the Narasimhan-Seshadri Theorem [12, 13], the moduli space further has a natural Kähler structure once a complex structure on  $\Sigma$  has been chosen. This gives a family of complex structures on the moduli space  $M$  parametrized by the Teichmüller space of  $\Sigma$ , which we denote  $\mathcal{T}$ . Consider now the trivial  $C^\infty(M, \mathcal{L}^k)$ -bundle  $\mathcal{H}^{(k)}$  over Teichmüller space  $\mathcal{T}$ . Then a

---

Supported in part by the center of excellence grant "Center for quantum geometry of Moduli Spaces" (DNRF95) from the Danish National Research Foundation.

Hitchin connection is a connection in  $\mathcal{H}^{(k)}$ , which preserve the sub-bundle of holomorphic sections  $H^{(k)}(M_\sigma, \mathcal{L}^k)$ ,  $\sigma \in \mathcal{T}$ . Further, we require it is given by adding a differential operator valued one form to the trivial connection in  $\mathcal{H}^{(k)}$

$$\nabla_V^H = \nabla_V^t + u(V),$$

for all vector fields  $V$  on  $\mathcal{T}$ . Hitchin found an explicit formula for  $u$ , which in [3] is proven to be given by the following global differential operator

$$u(V) = \frac{-1}{2n+4k} (\Delta_{G(V)} + 2\nabla_{G(V)dF_\sigma} + 4kV'[F]_\sigma).$$

Here  $F_\sigma$  is a Ricci potential for  $M_\sigma$  the moduli space with the Kähler structure given by the point  $\sigma \in \mathcal{T}$ . The notation  $V'$  indicate that we project  $V$  onto the holomorphic directions on  $\mathcal{T}$ . Finally the symmetric two tensor,  $G(V)$  is given by  $G(V) = V'[g_{M_\sigma}^{-1}]$  and the operator  $\Delta_{G(V)}$  is given by

$$\begin{aligned} \Delta_{G(V)} : C^\infty(M, \mathcal{L}^k) &\xrightarrow{\nabla_\sigma^{1,0}} C^\infty(M, T_\sigma^* \otimes \mathcal{L}^k) \xrightarrow{G \otimes Id} C^\infty(M, T_\sigma \otimes \mathcal{L}^k) \\ &\xrightarrow{\nabla_\sigma^{1,0} \otimes Id + Id \otimes \nabla_\sigma^{1,0}} C^\infty(M, T_\sigma^* \otimes T_\sigma \otimes \mathcal{L}^k) \xrightarrow{\text{tr}} C^\infty(M, \mathcal{L}^k) \end{aligned}$$

For this Hitchin connection it was shown in [2], that the curvature is given by

**Theorem 1.1 ( [2, Theorem 4.8])**

*The curvature of the Hitchin connection acts by*

$$F_\nabla^{2,0} = \frac{k}{(2k+2n)^2} P_k(\partial_{\mathcal{T}} c) \quad F_\nabla^{1,1} = \frac{ik}{2k+2n} (\theta - 2i\partial_{\mathcal{T}} \bar{\partial}_{\mathcal{T}} F) \quad F_\nabla^{0,2} = 0,$$

on sections of the bundle  $H^{(k)}$ .

Here  $\theta$  is as defined below in (1). The one form  $c$  on  $\mathcal{T}$  with values in  $C^\infty(M)$  is given by

$$c(V) = -\Delta_{G(V)} F - dFG(V)dF - 2nV'[F].$$

Finally,  $P_k(\partial_{\mathcal{T}} c(V, W))$  is the prequantum operator associated with the function  $\partial_{\mathcal{T}} c(V, W) \in C^\infty(M)$

$$P_k(\partial_{\mathcal{T}} c(V, W)) = \frac{i}{k} \nabla_{X_{\partial_{\mathcal{T}} c(V, W)}} + \partial_{\mathcal{T}} c(V, W),$$

where  $X_{\partial_{\mathcal{T}} c(V, W)}$  is the Hamiltonian vector field of the function  $\partial_{\mathcal{T}} c(V, W)$ . In fact it was observe in [2], that since the curvature must preserve the holomorphic sections  $X_{\partial_{\mathcal{T}} c(V, W)} = 0$  and so  $d_M(\partial_{\mathcal{T}} c(V, W)) = 0$ .

The form  $\theta$  is given as follows

$$\theta(\mu_1, \bar{\mu}_2) = \frac{1}{4} g_{\mathcal{M}_{VB}^{n,k}}(G(\mu_1) \omega_{\mathcal{M}_{VB}^{n,k}} \bar{G}(\bar{\mu}_2)), \quad (1)$$

In this paper we show that

**Lemma 1.2**

$$F_{\nabla^H}^{1,1} = \frac{ik}{2k+2n}(\theta - 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}F) = -\frac{ik(n^2-1)}{12(k+n)\pi}\omega_{\mathcal{T}}$$

And using this we can find a 1-form  $\tilde{c}$  on  $\mathcal{T}$  and we consider

$$\tilde{\nabla}^H = \nabla^H + \tilde{c} \otimes \text{Id}_{H^{(k)}}.$$

We remark that that it might be that  $\tilde{c}$  is zero. In any case after this (possible trivial) modification, we can prove that

**Theorem 1.3**

*The connection  $\tilde{\nabla}^H$  is still a Hitchin connection and has pure  $(1,1)$  curvature given by*

$$F_{\tilde{\nabla}^H} = \frac{ik(n^2-1)}{12(k+n)\pi}\omega_{\mathcal{T}}.$$

In section 2 we briefly recall our Kähler coordinate construction on the universal moduli space of vector bundles from [6]. In the following section 3, we compute the  $(1,1)$  part of the curvature of the Hitchin connection using the results of [6]. In final section 4 we modify the Hitchin connection by adding to it a scalar valued one-form on Teichmüller space tensor the identity of  $H^{(k)}$ , such that the resulting connection has only curvature of type  $(1,1)$ .

## 2 The Moduli Space of Vector Bundles

In order to compute the curvature of the Hitchin connection, we will use the local coordinates of [6], which we will now briefly recall. Let  $\Sigma$  be a surface of genus two or greater. Pick a point in  $\mathcal{T} \times M$ , that is a Riemann surface  $X$  and a holomorphic vector bundle  $E$  over it. For an element

$$\mu \oplus \nu \in H^1(X, TX) \oplus H^1(X, \text{End}E)$$

define a map

$$\chi^{\mu \oplus \nu} : \mathbb{H} \times \mathbf{SL}(n, \mathbb{C}) \rightarrow \mathbb{H} \times \mathbf{SL}(n, \mathbb{C})$$

which is annihilated by the following differential operator

$$\bar{\partial}_{\mathbb{H}}\chi^{\mu \oplus \nu} = (\mu - \frac{1}{2}\tilde{g}_X^{-1}\text{tr}\nu \otimes \nu) \cdot \partial_{\mathbb{H}}\chi^{\mu \oplus \nu} + \partial_{\mathbf{SL}(n, \mathbb{C})}\chi^{\mu \oplus \nu} \cdot \nu.$$

We will denote the projection to  $\mathbb{H}$  by  $\chi_1^{\mu \oplus \nu}$  and the projection to  $\mathbf{SL}(n, \mathbb{C})$  by  $\chi_2^{\mu \oplus \nu}$ .

The near by points contained in the coordinate neighbourhood in  $\mathcal{T} \times M$  are represented by a pair of equivalence classes of representations into

$\mathbf{PSL}(2, \mathbb{R})$  and  $\mathbf{SU}(n)$  respectively. Let's say our base point corresponds to  $\rho_{\mathbb{H}} : \pi_1(\Sigma) \rightarrow \mathbf{PSL}(2, \mathbb{R})$  and  $\rho_E : \pi_1(\Sigma - p) \rightarrow \mathbf{SU}(n)$ . Then the point corresponding to  $\mu \oplus \nu$  is

$$(\rho_{\mathbb{H}}^{\mu \oplus \nu}, \rho_E^{\mu \oplus \nu})(\gamma) = (\chi_1^{\mu \oplus \nu}(\rho_{\mathbb{H}}(\gamma)(\chi_1^{\mu \oplus \nu})^{-1}(z)), \chi_2^{\mu \oplus \nu}(\gamma z, e)\rho_E(\gamma)(\chi_2^{\mu \oplus \nu}(z, e))^{-1}).$$

We proved in [6] that this construction gives coordinates and moreover, we provided a Ricci potential for the total space and in particular, we showed in Theorem 4.2 in [6], that for the Ricci potential on  $M_{\sigma}$ , which is found in [15] fulfils

**Lemma 2.1**

For a pair of vector fields on  $\mathcal{T}$  represented by  $\mu_1$  and  $\bar{\mu}_2$  we have that

$$2\bar{\partial}_{\mathcal{T}}\partial_{\mathcal{T}}F(\mu_1, \bar{\mu}_2) = \text{tr}(\mu_1 P_{\text{End}E}^{1,0} \bar{\mu}_2 P_{\text{End}E}^{0,1}) - i \frac{n^2 - 1}{6\pi} \omega_{\mathcal{T}}(\mu_1, \bar{\mu}_2).$$

Where  $P_{\text{End}E}^{0,1}$  (resp.  $P_{\text{End}E}^{1,0}$ ) is the projection on harmonic  $(0, 1)$ -forms (resp.  $(1, 0)$ -forms) with values in  $\text{End}E$ .

### 3 The $(1, 1)$ -curvature of the Hitchin Connection

First we calculate  $G(V_{\mu})$  in coordiantes, here  $\mu$  denotes the betrami differential corresponding to  $V$  by the Kodaira-Spencer map. We recall from Hitchin [10] that  $G(V_{\mu})(\alpha, \beta) = \int_{\sigma} V'_{\mu}[-\star_{\sigma}] \text{tr} \alpha \otimes \beta$ . To calculate the variation of  $-\star_{\sigma}$ , we need to fix a harmonic 1-form,  $\nu$  on  $\Sigma$ . We split it into  $\nu = \nu_1 + \bar{\nu}_2^T$  at a point  $X \in \mathcal{T}$  where  $\nu_1, \nu_2$  are harmonic  $(0, 1)$ -forms on  $X$  with values in  $\text{End}E$ . Then we have that at a point  $(X_{\mu \oplus 0}, E)$ , we can use the quasiconformal maps  $\chi_1^{\mu \oplus 0}$  to change the complex structure on  $X$ , so that the complex structure on  $X_{\mu \oplus 0}$  is deccribed by a quotient construction of  $\mathbb{H}$  with the standard structure. Then  $\nu$  is given by

$$\begin{aligned} (\chi_1^{\mu \oplus 0})_*^{-1} \nu &= (\nu_1 (\overline{\partial \chi_1^{\mu \oplus 0}}) (d\bar{z} - \mu \frac{\partial \chi_1^{\mu \oplus 0}}{\partial \chi_1^{\mu \oplus 0}} dz) \\ &\quad + \nu_2 (\partial \chi_1^{\mu \oplus 0}) (-\bar{\mu} \frac{\overline{\partial \chi_1^{\mu \oplus 0}}}{\partial \chi_1^{\mu \oplus 0}} d\bar{z} + dz)) \circ (\chi_1^{\mu \oplus 0})^{-1}. \end{aligned}$$

So we can find the harmonic representativ of  $\nu$  at  $\mu$ , which we denote  $\nu^{\mu}$ , using the projections on harmonic  $(1, 0)$ -forms and  $(0, 1)$ -forms on  $X_{\mu \oplus 0}$  with values in  $\text{End}E$  to obtain that

$$\begin{aligned} \nu^{\mu} &= P_{\text{End}E}^{0,1} ((\overline{\partial \chi_1^{\mu \oplus 0}}) (\nu_1 d\bar{z} - \bar{\mu} \nu_2 d\bar{z})) \circ (\chi_1^{\mu \oplus 0})^{-1} \\ &\quad + P_{\text{End}E}^{1,0} ((\partial \chi_1^{\mu \oplus 0}) (\nu_2 dz - \mu \nu_1 dz)) \circ (\chi_1^{\mu \oplus 0})^{-1} \end{aligned}$$

Now  $I[\nu] = [-\star\nu^\mu]$  and as is seen in [10, Lemma 2.15] we have that  $V_\mu(I)[\nu] = [V_\mu(-\star\nu^\mu)]$ , since  $[-\star V_\mu\nu^\mu]$  is exact. To calculate  $V_\mu[-\star\nu^\mu]$ , we pull it back to  $X_0$  with  $\chi_1^{\mu\oplus 0}$  and find that

$$\begin{aligned} (\chi_1^{\mu\oplus 0})_*(-\star)\nu^\mu &= iP_{\text{End}E}^{0,1}((\overline{\partial\chi_1^{\mu\oplus 0}})(\nu_1 d\bar{z} - \bar{\mu}\nu_2 d\bar{z}))(-\mu(\partial\chi_1^{\mu\oplus 0})^{-1} + (\overline{\partial\chi_1^{\mu\oplus 0}})^{-1}) \\ &\quad - iP_{\text{End}E}^{1,0}((\partial\chi_1^{\mu\oplus 0})(\nu_2 dz + \mu\nu_1 dz))(\bar{\mu}(\overline{\partial\chi_1^{\mu\oplus 0}})^{-1} + (\partial\chi_1^{\mu\oplus 0})^{-1}). \end{aligned}$$

When we evaluate this at  $\varepsilon\mu$  and differentiate with respect to  $\varepsilon$ , then most of the terms have explicit factors of  $\varepsilon$  and are quickly seen to contribute  $-iP_{\text{End}E}^{1,0}\mu\nu_1 - i\mu P_{\text{End}E}^{0,1}\nu_1$ , at  $\varepsilon = 0$ . Now the only terms remaining are

$$P_{\text{End}E}^{0,1}((\overline{\partial\chi_1^{\mu\oplus 0}})(\nu_1 d\bar{z})(\overline{\partial\chi_1^{\mu\oplus 0}})^{-1})$$

and

$$P_{\text{End}E}^{1,0}((\partial\chi_1^{\mu\oplus 0})(\nu_2 dz)((\partial\chi_1^{\mu\oplus 0})^{-1})).$$

The harmonic projections are given as  $P_{\text{End}E}^{0,1} = I - \bar{\partial}\Delta_0^{-1}\bar{\partial}^*$  and  $P_{\text{End}E}^{1,0} = I - \partial\Delta_0^{-1}\partial^*$ . When we differentiated these with respect to  $\varepsilon$  the  $I$ 's will disappear and either the first or last  $\bar{\partial}$  or  $\bar{\partial}^*$  (resp.  $\partial$  or  $\partial^*$ ) in  $\bar{\partial}\Delta_0^{-1}\bar{\partial}^*$  (resp.  $\partial\Delta_0^{-1}\partial^*$ ) will not be differentiated. In the first case, we have an exact contribution, which does not change the cohomology class. In the second case the term will be zero, since  $\nu \in \ker \bar{\partial}^*$  ( $\bar{\nu}^T \in \ker \partial^*$ ). We now conclude that

$$V_\mu(I)[\nu] = [-iP_{\text{End}E}^{1,0}\mu\nu_1 - i\mu P_{\text{End}E}^{0,1}\nu_1].$$

And so we must have that

$$G(V_\mu)(\nu_1, \nu_2) = -2i \int_\Sigma \mu \text{tr} \nu_1 \nu_2,$$

and thus

$$G(V_{\bar{\mu}})(\bar{\nu}_1^T, \bar{\nu}_2^T) = 2i \int_\Sigma \bar{\mu} \text{tr} \bar{\nu}_1^T \bar{\nu}_2^T.$$

Now that we have an expression in our coordinates for  $G(V_\mu)$  at the center point, we can calculate (1) in local coordinates

$$\begin{aligned} G(V_{\mu_1})\omega_{\mathcal{M}_{VB}^{n,k}}\bar{G}(V_{\bar{\mu}_2})_{i\bar{j}} \\ = \left( \sum_{j,l} -2i \int_X \mu_1 \text{tr} \bar{\nu}_i^T \bar{\nu}_j^T (-I \int \text{tr} \nu_j \wedge \bar{\nu}_l^T) 2i \int_X \bar{\mu}_2 \text{tr} \nu_l \nu_k \right). \end{aligned}$$

Also recall that at the center point we have chosen our basis of  $\nu_i$ 's to be orthonormal and so  $P^{0,1}\alpha = -i \sum_i \nu_i \int_\Sigma \text{tr} \alpha \wedge \nu_i$  and so we obtain that

$$G(V_{\mu_1})\omega_{\mathcal{M}_{VB}^{n,k}}\bar{G}(V_{\bar{\mu}_2})_{i\bar{j}} = 4i \left( \int_X \mu_1 \text{tr} \bar{\nu}_i^T P^{1,0}(\bar{\mu}_2 \nu_j) \right).$$

Contract with the metric and using that  $\text{tr} P^{0,1} F = \sum_i \int_\Sigma (F \nu_i) \wedge \bar{\nu}_i^T$ , we get that

$$\theta(\mu_1, \bar{\mu}_2) = i \text{tr}(\mu_1 P^{0,1} \bar{\mu}_2 P^{1,0})$$

Thus by Lemma 2.1 and Theorem 1.1, we have proved Lemma 1.2.

## 4 Modification of $(2, 0)$ -part of the Curvature

In this section we prove Theorem 1.3. First we observe that by the result of the previous section we can use the Bianchi identity for the curvature to conclude that the  $(2, 0)$ -part of the curvature of the Hitchin connection is  $\bar{\partial}_{\mathcal{T}}$  closed, and hence  $d_{\mathcal{T}}$  closed by the following argument. We let  $V', W'$  be holomorphic vector fields on  $\mathcal{T}$  and  $U''$  anti-holomorphic. Then the Bianchi identity gives

$$0 = U''(F^{2,0}(V', W')) - V'(F^{1,1}(W', U'')) + W'(F^{1,1}(U'', V')).$$

But since  $F^{1,1}$  is proportional to the symplectic form on  $\mathcal{T}$ , we get that

$$-V'(F^{1,1}(W', U'')) + W'(F^{1,1}(U'', V')) = \partial_{\mathcal{T}} F^{1,1}(V', W', U'') = 0.$$

We conclude that  $\bar{\partial}_{\mathcal{T}} F^{2,0} = 0$ . Finally we recall from [2] that  $d_M F^{2,0} = 0$  as well. Now use that  $F^{2,0}$  is mapping class group invariant, so it pushes down to a closed  $(2, 0)$ -form on the moduli space  $\mathcal{M}_g$  of genus  $g$  curves.

To proceed further we need to assume that  $\Sigma$  has genus three or greater, since this assumption will imply that the following two statements are true.

- The moduli space of genus  $g \geq 3$  curves,  $\mathcal{M}_g$ , contains complete curves. This means that there exist a complex surface  $S$  and a holomorphic embedding  $S \rightarrow \mathcal{M}_g$ . For explicit construction see [19] for genus 3 and for higher genus references there in.
- The second thing we need is Harer's result [9], that for  $g \geq 3$  the second cellular homology is

$$H_2(M_g, \mathbb{C}) \cong \mathbb{C}.$$

Harer's result implies that  $H_{dR}^2(\mathcal{M}_g, \mathbb{C}) \cong \mathbb{C}$ , since it is dual to  $H_2(M_g, \mathbb{C})$ . We know that the generator must be  $\omega_{\mathcal{T}}$ , thus in order to prove that  $F^{(2,0)}$  is exact, we need to show that its class is 0. We can use the Surface  $S$ , which is a complex embedding submanifold and we can integrate  $F^{(2,0)}$  over it and as it is a  $(2, 0)$ -form the result is 0, at the same time we know that the integral of  $\omega_{\mathcal{T}}$  is non-zero over  $S$  and so the cohomology class of  $F^{(2,0)}$  is 0. This means that there exists a 1-form  $\tilde{c}$  on  $\mathcal{M}_g$  such that  $F^{(2,0)} = -d_{\mathcal{M}_g} \tilde{c}$ .

Now we can pull back  $\tilde{c}$  to  $\mathcal{T}$  and then define a slightly modified, but still mapping class group invariant Hitchin connection, as discussed in the

introduction. We just need to check that it is still a Hitchin connection. By [3, Lemma 2.2] it is enough to prove that

$$\frac{i}{2}V[I](\nabla_V^t)^{1,0}s + \nabla_{M_\sigma}^{0,1}(u(V) + \tilde{c}(V))s = 0$$

But since  $\nabla^t + u(V)$  is a Hitchin connection, this reduce to showing  $\bar{\partial}_{M_\sigma}\tilde{c}(W, V) = 0$ . But that follows from the defining identity, since  $d_{\mathcal{T}}\tilde{c} = \partial_{\mathcal{T}}c$  which is a  $(2, 0)$  from. To calculate the curvature we see that

$$F_{\tilde{\nabla}}(V, W) = [\nabla_V + \tilde{c}(V), \nabla_W + \tilde{c}(W)] = [\nabla_V, \nabla_W] + [\tilde{c}(V), \nabla_W] + [\nabla_V, \tilde{c}(W)] + [\tilde{c}(V), \tilde{c}(W)].$$

The first term is just the curvature calculated in Theorem 1.1. The two next terms only contribute  $-W[\tilde{c}(V)] + V[\tilde{c}(W)] = d_{\mathcal{T}}\tilde{c}(W, V)$ , since  $\tilde{c}$  does not depend on where we are in the moduli space of vector bundles and so commute with the differential operator  $u$ . The last term is also zero, since multiplication by functions commute, hence we conclude that

$$F_{\tilde{\nabla}}(V, W) = \frac{(n^2 - 1)k}{6\pi(k + n)}\omega_{\mathcal{T}}(V, W) + F_{\nabla}^{(2,0)}(V, W) + d_{\mathcal{T}}\tilde{c}(V, W) = \frac{(n^2 - 1)k}{6\pi(k + n)}\omega_{\mathcal{T}}(V, W)$$

where the last equality follows by the construction of  $\tilde{c}$ , since  $F_{\nabla}^{(2,0)}(V, W) = -d_{\mathcal{T}}\tilde{c}(V, W)$ . This concludes the proof of Theorem 1.3.

## References

- [1] L. Ahlfors & L. Bers (1960). Riemann's mapping theorem for variable metrics. *Annals of Mathematics*, 72(2):pp. 385–404.
- [2] J. E. Andersen & N. L. Gammelgaard (2011). Hitchin's projectively flat connection, Toeplitz operators and the asymptotic expansion of TQFT curve operators. In *Grassmannians, moduli spaces and vector bundles*, volume 14 of *Clay Math. Proc.*, pages 1–24. Amer. Math. Soc., Providence, RI.
- [3] J. E. Andersen (2012). Hitchin's connection, Toeplitz operators, and symmetry invariant deformation quantization. *Quantum Topol.*, 3(3-4):293–325.
- [4] J. E. Andersen, N. L. Gammelgaard & M. R. Lauridsen (2012). Hitchin's connection in metaplectic quantization. *Quantum Topol.*, 3(3-4):327–357.
- [5] J. E. Andersen & N. S. Poulsen (2016). Coordinates for the Universal Moduli Space of Holomorphic Vector Bundles *arXiv:1603.00294*.
- [6] J. E. Andersen & N. S. Poulsen (2016). An explicit Ricci potential for the Universal Moduli Space Vector Bundles *arXiv:1609.xxxx*.

- [7] S. Axelrod, S. Della Pietra, E. Witten, "Geometric quantization of Chern Simons gauge theory.", *J.Diff.Geom.* **33** (1991) 787–902.
- [8] D.S. Freed, "Classical Chern-Simons Theory, Part 1", *Adv. Math.* **113** (1995), 237–303.
- [9] Harer, John(1983), The second homology group of the mapping class group of an orientable surface, *Invent. Math.*,72(2):221–239,
- [10] N. J. Hitchin(1990). Flat connections and geometric quantization. *Comm. Math. Phys.*, 131(2):347–380.
- [11] V. B. Mehta & C. S. Seshadri(1980). Moduli of vector bundles on curves with parabolic structures, *Math. Ann.*, 248,(3):205–239
- [12] M.S. Narasimhan and C.S. Seshadri, "Holomorphic vector bundles on a compact Riemann surface", *Math. Ann.* **155** (1964) 69 – 80.
- [13] M.S. Narasimhan and C.S. Seshadri, "Stable and unitary vector bundles on a compact Riemann surface", *Ann. Math.* **82** (1965) 540 – 67.
- [14] M. S. Narasimhan, R. R. Simha, R. Narasimhan & C. S. Seshadri (1963). *Riemann surfaces*, volume 1 of *Mathematical Pamphlets*. Tata Institute of Fundamental Research, Bombay.
- [15] L. A. Takhtadzhyan & P. G. Zograf(1989). The geometry of moduli spaces of vector bundles over a Riemann surface. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(4):753–770, 911.
- [16] L. A. Takhtajan & P. Zograf (2008). The first Chern form on moduli of parabolic bundles. *Math. Ann.*, 341(1):113–135.
- [17] L. A. Takhtajan & P. G. Zograf(1991). A local index theorem for families of  $\bar{\partial}$ -operators on punctured Riemann surfaces and a new Kähler metric on their moduli spaces. *Comm. Math. Phys.*, 137(2):399–426.
- [18] L. A. Takhtajan & P. G. Zograf (1987). A local index theorem for families of  $\bar{\partial}$ -operators on Riemann surface. *Usp.Mat. Nauk*, 42(6):169-190(in Russian);*Russ. Math. Surv.* 42(6) 169-190 (1987).
- [19] Chris Zaal (1995) Explicit complete curves in the moduli space of curves of genus three, *Geom. Dedicata*, 56(2):185–196,
- [20] S. A. Wolpert (1986). Chern forms and the Riemann tensor for the moduli space of curves *Invent. Math.*, 85(1):119–145.